MODELS FOR THE LOCATION-ALLOCATION-PROBLEM IN URBAN AND REGIONAL INFRASTRUCTURE PLANNING

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1. Introduction

Many of the infrastructure facilities that are considered to be necessary prerequisites for the use of an urban or rural region by other activities can be considered 'central facilities'. This concept is derived from the notion that a spatially distributed demand for services provided by these facilities can be offered only at fewer locations compared to the many locations of demand. Therefore, these facilities are supposed to be located in such a way that they are central to the locations of demand. Infrastructure facilities to which this essential attribute of central location applies range from kindergartens, schools, or hospitals to emergency call boxes, mail boxes, or telephone booths to doctors' offices, pharmacies, or retail centers to waste disposal sites, fire stations, or fall-out shelters. These are only a few examples of the many facilities that are regarded as infrastructure facilities demanding a central location. They all cut across the well accepted categories of private sector and public sector facilities. Though some of the infrastructure facilities may be attributed to one or the other of these two categories without dispute their private or public ownership becomes less important once the public, politicians and planners conceive them as one activity system among other activity systems such as residential or industrial activity systems. Furthermore, when facility location is guided by public means and measures within the overall development of interrelated activity systems of an urban or regional setting, ownership is not a decisive criterion.

During the past decade the theoretically oriented discussion on central facilities locations has been dominated by the distinction between public central facilities and private central facilities.(1) Related to the ownership of central facilities this differentiation is helpful in identifying inherent characteristics of processes of decision making in the private versus the public realm. It is, however, of minor importance with respect to the subject of locational decisions once each central facility is understood to be part of an activity system, a subsystem itself of an overall activity system. With this perspective in mind the subject of locational decisions implies two different but interdependent issues: firstly, the direct decision on a location proper and, secondly, the indirect decision on ongoing processes of spatial-functional interactions. The first issue of this complex decision has been analyzed quite clearly with regard to its implications in the context of economic and social externalities. The second issue has usually not been addressed explicitly but has rather been covered either by somewhat nebulous concepts of accessibility, social utility, goods-oriented versus consumer-oriented or ordinary versus extraordinary public services or by a simultaneous consideration of transportation costs and facility investment costs depending on the number of facilities and the sites needed for locating them. With respect to many public as well as private infrastructure facilities the consideration of trade-offs between investment and transportation costs is misleading for either one of two reasons: In the case of many central facilities the costs of investment and the costs of transportation are split and paid for by two different groups: the 'suppliers', i.e. those who invest, and the 'users', i.e. those who utilize the central services provided via the central facilities. Users pay either directly for their expenses when they travel to the central facilities or they pay indirectly for transportation costs by paying taxes and fees. For many other central facilities, especially emergency service facilities, the combined consideration of investment and transportation costs is logically impossible or altogether irrelevant. Instead of integrating different and to some extent conflicting locational criteria into one overall objective it may be more appropriate to pursue an approach that tries to analyze the trade-offs that have to be acceptable when preferring one objective against another objective.[(2)]

Instead of arguing along the private versus public dichotomy it is suggested to neglect these categories in the context of defining locations for central facilities in favor of reconsidering the notion of 'central location'. For the purpose of this analysis 'central location' is regarded as the dominant locational factor. It is intended to develop some basic operational definitions providing the basis for a series of models for the location-allocation-problem.

'Central location' is understood to be the result of a locational decision process and is understood to be realized during the process of plan implementation. The specific meaning of 'central location' is derived from the spatial interrelationships that manifest themselves in ongoing hourly, daily, weekly trip patterns or some other periodic system of trips in space. The evaluation of the attributes of these spatial interrelationships in terms of planning goals results in different concepts for 'central location'.

The deliberate restriction on the locational factor 'central location' allows one to concentrate on the spatial behavioral patterns that are an expression of performed spatial-functional interrelationships. It allows one as well, to concentrate on the intended influencing of all those involved in the locational decision process, an influencing that is directed towards the spatial-functional interrelationships. This influencing is interpreted as a 'locational goal'. Spatial behavioral patterns and locational goals form the starting points for operational definitions of 'central location'. They allow, too, the set-up of a systematic substantive framework for a series of models for the location-allocation-problem.

Once this substantive-oriented framework for the location-allocation-problem has been established the technical-oriented approach to model systemization becomes a useful concept.(3) This technical-oriented approach deals with dimensions of models such as static versus dynamic, deterministic versus stochastic, normative versus explanative, or location on a plane versus location on a network.(4) This approach to model stratification commonly used in the context of models for the location-allocation-problem is irrelevant to the issue of 'central location'. It is, however, a valuable tool in characterizing the planning situation as defined by the available data base on the planning region. It is, as well, a valuable tool in characterizing the type of model in terms of modelling theory. Thus, the technical-oriented framework may be used in choosing the method of solution being appropriate to a model for the location-allocation-problem.

The two basic issues, the one of spatial behavioral patterns and the one of locational goals, are discussed in Sections 2 and 3. Section 4 proposes definitions for the locational factor of 'central location' being based on spatial behavioral patterns and locational goals. These definitions in connection with definitions for districts, i.e. service areas of central facilities, provide a framework for a series of basic models for the location-allocation-problem. These models are described in Section 5 and their mathematical programming formulation is presented in the Appendix.

2. Concepts for Spatial Behavioral Patterns

Compared to the broad range of different types of central facilities there exist only few empirical studies on spatial behavioral patterns that can shed light on spatio-functional relationships between central facilities and users. Central facilities for which empirical studies have been published are generally limited to retail facilities, warehouses and production plants, and health services facilities.

With respect to retail facilities the concept of potential is the explanatory concept for the spatial behavioral pattern. According to this concept the spatial behavioral pattern of consumers as expressed by the choice of one particular facility out of all facilities is determined by the attractiveness of the retail facility as well as by the spatial distance between the facility and the location of the consumer.[(5)] With respect to warehouses and production plants the distribution of goods and products or the shipment of materials needed at the production plant is steered by the criteria of transportation costs. In this case the concept of a spatial behavioral pattern is based on the minimization of transportation costs. With respect to health services empirical studies have proven that spatial distances influence the frequency of health services requested by patients.[(6)]

These three examples suggest the following three principles underlying spatial patterns:

- The principle of minimizing spatial friction without an influence on the frequency of utilization of a system of central facilities; (7)

- The principle of minimizing spatial friction with an influence on the frequency of utilization of a system of central facilities;

- The principle of maximizing potential.

It is assumed that one of these principles determines the spatial behavioral pattern of either a user or a supplier. The user has to be considered when he has to pay for surmounting spatial friction, and the supplier has to be considered when he has to pay for surmounting spatial friction. Starting from the assumption that all users have to be taken into consideration such that the demand of each user for central services is covered, three different spatial behavioral patterns of users and one spatial behavioral pattern of suppliers can be derived. These patterns are:

- Spatial behavioral pattern of users based on the principle of minimizing spatial friction without influence on the frequency of utilization of a system of central facilities:

Each user goes to that facility of a system of central facilities located next to his own location. The frequency of utilization is not influenced by the spatial friction, i.e. it is not decreasing with increasing spatial friction. A spatial behavioral pattern such as this one implies that the user is able to determine which facility is nearest to him in terms of metrical distance, time distance, costs of transportation or a distance concept based on physical and psychological effort. Examples of systems of central facilities for which this concept of a spatial behavioral pattern is appropriate are post offices, emergency call boxes, train stations, or mass transit stops.

- Spatial behavioral pattern of users based on the principle of minimizing spatial friction with influence on the frequency of utilization of a system of central facilities:

Each user goes to that facility of a system of central facilities located next to his own location as he wants to maximize the frequency of utilization of services provided by the system of facilities. The frequency of utilization decreases with increasing friction of space. A spatial behavioral pattern such as this one implies that with respect to his frequency of utilization a user is influenced by the spatial friction he has to surmount. It is implied, too, that the individual user is able to evaluate the spatial friction between his location and the locations to all central facilities enabling him to determine the nearest central facility as the one he wants to go to. The amount of utilization is solely influenced by the spatial friction and not by the attractivity of the central facilities. Examples of systems of central facilities for which this concept of a spatial behavioral pattern is appropriate are health service facilities such as hospitals and health care centers or leisure facilities, play-grounds, and sports facilities.

- Spatial behavioral pattern of users based on the principle of maximizing potential:

Each user goes to that facility of a system of central facilities that exerts the highest potential at this location. The potential of a facility at a user location results from its attractivity and the friction of space considered to be a factor of resistance. A spatial behavioral pattern such as this one implies that the user is able to evaluate and rate, firstly, the attractivity of each of the central facilities and, secondly, to determine the amount of spatial friction between his location and the locations of all the central facilities. It is assumed that the amount of demand for central services for each user is neither increased nor decreased by the attractiveness or the spatial frictions. Examples of central facilities systems for which this concept of a spatial behavioral pattern is appropriate are retail centers. Furthermore, this concept may be appropriate, too, for facilities such as libraries, neighborhood centers for senior citizens, facilities for teen-agers or leisure facilities.

- Spatial behavioral pattern of a supplier or a group of suppliers based on the principle of minimizing spatial friction without influence on the frequency of utilization of a system of central facilities:

Each user is served by a supplier or a group of suppliers via that facility of a system of central facilities which is located next to the user location. The amount of service is not influenced by the amount of spatial friction, i.e. it is not decreasing with increasing spatial friction. A spatial behavioral pattern such as this one implies that the supplier or the group of suppliers is able to determine the nearest central facility for each user location. Examples of systems of central facilities for which this concept of a spatial behavioral pattern is appropriate are warehouses, fire-stations or ambulance dispatch centers.

3. Political and Societal Goals for Locational Patterns of Central Facilities

In the context of locational decisions for systems of central facilities the foremost goal - in spatial terms - is to assure the availability of central services by an appropriate locational pattern of these facilities. Thus, the locational pattern has to guarantee a 'spatial availability'. Criteria for measuring spatial availability have to be derived from a consideration of spatial behavioral patterns as well as from a rationale underlying the political and/or societal goals for locational patterns of systems of central facilities.

Starting from the point of view of spatial behavioral patterns the spatial availability of a system of central facilities has to be measured by either one of the following three criteria:

- Cost of surmounting spatial friction:

The cost of surmounting spatial friction is the appropriate criterion for such systems of central facilities for which the principle of minimizing spatial friction without an influence on the frequency of utilization is appropriate with respect to the spatial behavioral pattern either of users or of suppliers.

- Amount of utilization of central services:

The amount of utilization of central services provided via facilities is the applicable criterion for such systems of central facilities for which the principle of minimizing spatial friction with an influence on the frequency of utilization is appropriate with respect to the spatial behavioral pattern of users.

- Access opportunity for users:

The criterion of access opportunity for users to central services applies for such systems of central facilities for which the principle of maximizing potential is appropriate with respect to the spatial behavioral pattern of users.

The spatial availability of a system of central facilities can be measured in either one of the following two ways depending on the locational goal pursued: Firstly, the spatial availability can be measured by looking at the totality of spatio-functional relationships existing between all users and the system of central facilities. Secondly, it can be measured by looking at the spatio-functional relationship existing between each individual user and the system of central facilities in comparison to all users or between each individual facility and all users in comparison to all facilities of the system of central facilities.

In the first case, the criteria for spatial friction are:

- the sum of costs of surmounting spatial friction;
- the total amount of utilization of all central facilities;
- the total amount of access opportunity for the sum of all user locations.

In the second case, the criteria for spatial availability are:

- the equalized costs of surmounting spatial friction with respect to each individual user;
- the equalized amount of utilization with respect to each individual facility of the system of central facilities;
- the equalized amount of access opportunity with respect to all user locations.

These six criteria imply a differentiation with regard to a collective locational rationale versus an equalizing locational rationale. By introducing this differentiation it is intended to delimit three spheres of locational choices. Each one of these three spheres is based on one particular principle of spatial behavioral pattern. Furthermore, this differentiation allows the evaluation of locational patterns of systems of central facilities with respect to locational efficiency versus locational equity.[8] The three spheres of locational choices are indicated in Figure 2.

4. Definitions for Central Locations and Districts

Using the principles of spatial behavioral patterns as well as the locational goals as points of relevance altogether nine definitions are derived for central locations and four definitions for districts, i.e. service areas, of facilities located at central locations (See Figures 1 and 2).

The definitions for the central locations are:(9)

Model S-1: Central Point:

The point minimizing the sum of weighted distances;

Model S-2: Median Point:

The point minimizing the sum of weighted rectangular distances;

Model S-3: Arithmetic Mean Point:

The point minimizing the sum of weighted squared distances;

Model S-4: Center Point:

The point minimizing the maximum unweighted distance;

Model S-5: Facility-Oriented Point of Potential:

The point maximizing facility utilization;

Model S-6: User-Oriented Point of Potential:

The point maximizing access opportunity for the sum of user locations;

Model S-7: User-Oriented Point of Equalized Potential:

The point equalizing access opportunity for the individual user location with regard to all other user locations;

Model S-8: Radial Point:

The point from which each user location can be reached within a maximum allowable distance;

Model S-9: Constrained Radial Point:

The point from which the maximum number of users can be reached within a maximum allowable distance.

The definitions for the districts or their boundaries, respectively, are:[10]

Model E-1: Districts delimited by boundaries formed by points of equal distances between neighboring central facilities;

Model E-2: Districts delimited by boundaries formed by points of equal potential between neighboring central facilities;

Model E-3: Districts as attractivity-dependent and distance-dependent spheres of influence of central facilities;

Model E-4: Districts delimited by boundaries formed by points of a maximum allowable distance.

5. A Systematic of Models for the Location-Allocation-Problem

Based on the nine models for the location-problem and the four models for the allocation-problem, three groups of altogether ten models for the location-allocation-problem are derived. Each one of these models can be traced back to one particular principle of spatial behavioral pattern and can be attached to one of the three spheres of locational choices as shown in Figure 2. Mathematical programming formulations for these ten models are presented in the Appendix:[11]

Model SE-1: Locations and districts for central points;

Model SE-2: Locations and districts for center points;

Model SE-3: Locations and districts for facility-oriented points of potential;

Model SE-3/Variant 1: Locations and districts for facility-oriented points of equalized potential;

Model SE-3/Variant 2: Locations and spheres of influence for facility-oriented points of equalized potential;

Model SE-4: Locations and districts for user-oriented points of potential;

Model SE-5: Locations and districts for user-oriented points of equalized potential;

Model SE-6: Locations and spheres of influence for user-oriented points of equalized potential;

Model SE-7: Locations and districts for radial points;

Model SE-8: Locations and districts for constrained radial points.

The models SE-1 to SE-8 as presented here and in the Appendix have to be regarded as basic versions. The only locational factor that is represented by these models is the one of 'central location'. Though this locational factor is considered to be a dominating one in the context of locational decisions for systems of central facilities an extension of the basic models may be desirable. This can be done primarily along two lines: Firstly, the models may be extended by including those other locational factors which are to some degree directly related to the central location of facilities. Locational factors that could be thought of in this connection are:(12)

- minimal and maximal capacities of central facilities have to be observed;
- a maximum allowable distance between central locations and user locations must not be surpassed; (13)
- already existing central facilities have to be integrated into the new, extended system of central facilities;
- functional and spatial-functional, hierarchical structures of systems of central facilities must be represented.

Secondly, the models may be extended by including time-dependent dynamic shifts of the supply-component and/or the demand-component of systems of central facilities. Up till now, the implicit assumption has been made that all facilities of a system of central facilities will be in existence at the same point of time. This assumption should be dropped for two reasons. On the one hand, the implementation of a total system of central facilities depends on the availability of financial, man-power, and spatial-physical resources. This implies that a system of central facilities can usually be implemented only step by step. On the other hand, a phased implementation of a system of central facilities might be necessitated by spatial and/or qualitative shifts of the demand for central services.

In order to handle time-dependent changes caused by changing restrictions of resources or by changes of demand, four different strategies of implementation can be suggested. These strategies differ from each other by the uncertainty about future levels of demand as well as by relative independence of each implementation phase within the sequence of phases. These four strategies are:[14] the strategy of development phases, the strategy of a development sequence, the strategy of completion phases, and the strategy of a completion sequence. In the differentiation of 'development' and 'completion' the attempt is made to give consideration to the growth of a system of central facilities out of preceding time phases in the first case, and its gradual completion towards a predetermined final state in the latter case.[15]

6. Appendix: Mathematical Programming Formulations for Models SE-1 to SE-8

The following notations are used throughout the Appendix:

d_{ij} distance from the i-th central location to the j-th user location

d_{ji} distance from the j-th user location to the i-th central location

d^+ maximum allowable distance between a central location and a user location

β distance exponent for handling a linear relationship (β=1) or a non-linear relationship ($0<\beta<1$ or $\beta>1$) between distances and transportation costs

m number of central locations

c_i attractiveness of the facility at the i-th central location

n number of user locations

r_j number of users located at the j-th user location

α_{ij}, α_{ji} 0-1-variable (allocation coefficient), where:

$\left.\begin{array}{l}\alpha_{ij} = 1 \\ \alpha_{ji} = 1\end{array}\right\}$ if the j-th user location is within the service area (district) of the facility at the i-th central location and

$\left.\begin{array}{l}\alpha_{ij} = 0 \\ \alpha_{ji} = 0\end{array}\right\}$ otherwise

b number of given potential central locations

t number of possibilities to form different combinations containing m central locations out of b potential central locations

d'_{ji} distance from the j-th user location to the i-th central location which is identical with one of the b given potential central locations

α'_{ji} 0-1-variable (allocation coefficient), where:

$\alpha'_{ji} = 1$ if the j-th user location is within the service area (district) of the facility at the i-th central location which is identical with one of the b given potential central locations and

$\alpha'_{ji} = 0$ otherwise

${}_{i}I_{j}$ amount of utilization of a facility located at the i-th central location by all users located at the j-th user location

${}_{i}I$ amount of utilization of a facility located at the i-th central location by all users located at all n user locations

${}_{i}\Phi_{j}$ probability indicating that a user located at the j-th user location will utilize central services offered at the i-th central location

${}_{j}I_{i}$ access-opportunity at the j-th user location with regard to the facility at the i-th central location

$\hat{S}$ standard deviation of the amounts of utilization of all m central facilities

$\hat{S}$ is calculated by applying the following formula for the standard deviation S of u observations v_w:

$$S = ((u\ (u-1))^{-1}\ (u \sum_{w=1}^{u} v_w^2 - (\sum_{w=1}^{u} v_w)^2\ ))^{1/2}$$

X_i 0-1-variable representing potential central locations, where:

$X_i = 1$ if a central facility is located at X_i, and

$X_i = 0$ otherwise

MODEL SE-1: LOCATIONS AND DISTRICTS FOR CENTRAL POINTS

The locations and districts of m central facilities have to be determined in such a way that: (a) each user is allocated to exactly one central facility and (b) the sum of weighted distances is a minimum:

Minimize:

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} r_j\ d_{ij}^{\beta}\ \alpha_{ij} \qquad \text{....(1.1)}$$

subject to:

$$\alpha_{ij} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{for all } i,\ i=1,\dots,m;\ \text{for all } j,\ j=1,\dots,n \qquad \text{....(1.2)}$$

$$\sum_{i=1}^{m} \alpha_{ij} = 1 \quad \text{for all } j,\ j=1,\dots,n \qquad \text{....(1.3)}$$

$$d_{ij} \geq 0 \quad \text{for all } i,\ i=1,\dots,m;\ \text{for all } j,\ j=1,\dots,n \qquad \text{....(1.4)}$$

MODEL SE-2: LOCATIONS AND DISTRICTS FOR CENTER POINTS

The locations and districts of m central facilities have to be determined in such a way that: (a) each user is allocated to at least one central facility and (b) the maximum distance is minimized that exists in any of the m districts between the central facility and the user locations allocated to it:

Minimize:

$$Z = \max_{j} \min_{i} d_{ij}^{\beta} \alpha_{ij} \qquad \text{....(2.1)}$$

subject to:

$$\alpha_{ij} = \begin{cases} 0 \\ 1 \end{cases} \qquad \text{for all } i,\ i=1,\dots,m \quad \text{for all } j,\ j=1,\dots,n \qquad \text{....(2.2)}$$

$$\sum_{i=1}^{m} \alpha_{ij} \geq 1 \qquad \text{for all } j,\ j=1,\dots,n \qquad \text{....(2.3)}$$

$$d_{ij} \geq 0 \qquad \text{for all } i,\ i=1,\dots,m \quad \text{for all } j,\ j=1,\dots,n \qquad \text{....(2.4)}$$

MODEL SE-3: LOCATIONS AND DISTRICTS FOR FACILITY-ORIENTED POINTS OF POTENTIAL

The locations and districts of m central facilities have to be determined in such a way that: (a) each user is allocated to exactly one central facility and (b) the utilization of all central facilities is maximized:

Maximize:

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} {}_{i}I_{j} \, \alpha_{ji} \qquad \text{....(3.1)}$$

subject to:

$$\alpha_{ji} = \begin{cases} 0 \\ 1 \end{cases} \qquad \text{for all } i,\ i=1,\dots,m \quad \text{for all } j,\ j=1,\dots,n \qquad \text{....(3.2)}$$

$$\sum_{i=1}^{m} \alpha_{ji} = 1 \qquad \text{for all } j,\ j=1,\dots,n \qquad \text{....(3.3)}$$

$$d_{ji} \geq 0 \qquad \text{for all } i,\ i=1,\dots,m \quad \text{for all } j,\ j=1,\dots,n \qquad \text{....(3.4)}$$

where:

$${}_{i}I_{j} = r_{j} \,/\, (1 + d_{ji}^{\beta}) \qquad \text{....(3.5)}$$

MODEL SE-3 / VARIANT 1: LOCATIONS AND DISTRICTS FOR FACILITY-ORIENTED POINTS OF EQUALIZED POTENTIAL

The locations and districts of m central facilities are to be determined in such a way that: (a) each user is allocated to exactly one central facility, (b) the amount of utilization is as equal as possible when comparing the central facilities with each other, and (c) the central facilities are located on potential central locations that are given:

Out of the set of b potential central locations altogether t combinations with m central locations can be formed:

$$t = \frac{b!}{(b-m)!\ m!} \qquad \text{....(4.0)}$$

For each one of the t combinations the following objective function can be determined so that the allocation coefficients α'_{ji} are known:

Maximize:

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} {}_iI_j \ \alpha_{ji} \qquad \text{....(4.1)}$$

subject to:

$$\alpha_{ji} = \begin{cases} 0 \\ 1 \end{cases} \qquad \begin{array}{l} \text{for all } i,\ i=1,\ldots,m \\ \text{for all } j,\ j=1,\ldots,n \end{array} \qquad \text{....(4.2)}$$

$$\sum_{i=1}^{m} \alpha_{ji} = 1 \qquad \text{for all } j,\ j=1,\ldots,n \qquad \text{....(4.3)}$$

$$d_{ji} = d'_{ji} \qquad \begin{array}{l} \text{for all } i,\ i=1,\ldots,m \\ \text{for all } j,\ j=1,\ldots,n \end{array} \qquad \text{....(4.4)}$$

In a next round of calculations for each one of the t combinations the standard deviation $\hat{S}$ is calculated:

$$\hat{S} = ((m\ (m-1))^{-1}(m \sum_{i=1}^{m} (\sum_{j=1}^{n} {}_iI_j\ \alpha_{ji})^2 - (\sum_{i=1}^{m} \sum_{j=1}^{n} {}_iI_j\ \alpha_{ji})^2\ ))^{1/2} \qquad \text{....(4.5)}$$

where:

$$\alpha_{ji} = \alpha'_{ji} \qquad \begin{array}{l} \text{for all } i,\ i=1,\ldots,m \\ \text{for all } j,\ j=1,\ldots,n \end{array}$$

and

$$d_{ji} = d'_{ji} \qquad \begin{array}{l} \text{for all } i,\ i=1,\ldots,m \\ \text{for all } j,\ j=1,\ldots,n \end{array}$$

In equations (4.1) and (4.5) equation (3.5) is used to calculate ${}_iI_j$.

That combination having the minimum value for $\hat{S}$ is the optimal solution looked for.

MODEL SE-3 / VARIANT 2: LOCATIONS AND SPHERES OF INFLUENCE FOR FACILITY-ORIENTED POINTS OF EQUALIZED POTENTIAL

The locations and spheres of influence of m central facilities have to be determined in such a way that: (a) the total demand of each user is allocated proportionally to all central facilities according to a probability which considers the attractiveness of and the distance to each of the central facilities, (b) the amount of utilization at the central facilities is as equal as possible when compared with each other, and (c) the central facilities are located on potential central locations that are given:

According to equation (4.0) t combinations with m central locations can be formed out of a set of b potential central locations. For each one of the t combinations the following equations (5.1), (5.2), and (5.3) are calculated:

- the sum of utilization at the i-th central facility:

$$ {}_iI = \sum_{j=1}^{n} \cdot r_j \quad {}_i\Phi_j \qquad \text{....(5.1)} $$

- with ${}_i\Phi_j$ being a probability factor:

$$ {}_i\Phi_j = c_i \ / \ (1+d_{ji}^{\beta}) \ \left(\sum_{i=1}^{m} c_i \ / \ (1+d_{ji}^{\beta}) \right) \qquad \text{....(5.2)} $$

- the standard deviation $\hat{S}$ of the sums of utilization:

$$ \hat{S} = \left((m\ (m-1))^{-1} \left(m \sum_{i=1}^{m} {}_iI^2 \ - \ \left(\sum_{i=1}^{m} {}_iI \right)^2 \right) \right)^{1/2} \qquad \text{....(5.3)} $$

The combination having the minimum value for $\hat{S}$ is the optimal solution looked for.

MODEL SE-4: LOCATIONS AND DISTRICTS FOR USER-ORIENTED POINTS OF POTENTIAL

The locations and districts of m central facilities have to be determined in such a way that: (a) each user is allocated to exactly one central facility and (b) the sum of access-opportunities of all user locations is a maximum:

Maximize:

$$ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} {}_jI_i \quad \alpha_{ji} \qquad \text{....(6.1)} $$

subject to:

$$ \alpha_{ji} = \begin{cases} 0 \\ 1 \end{cases} \qquad \text{for all } i,\ i=1,\dots,m \quad \text{for all } j,\ j=1,\dots,n \qquad \text{....(6.2)} $$

$$ \sum_{i=1}^{m} \alpha_{ji} = 1 \qquad \text{for all } j,\ j=1,\dots,n \qquad \text{....(6.3)} $$

$$ d_{ji} \geq 0 \qquad \text{for all } i,\ i=1,\dots,m \quad \text{for all } j,\ j=1,\dots,n \qquad \text{....(6.4)} $$

where:

$$_jI_i = c_i \;/\; (1 + d_{ji}^{\beta}) \qquad \ldots.(6.5)$$

MODEL SE-5: LOCATIONS AND DISTRICTS FOR USER-ORIENTED POINTS OF EQUALIZED POTENTIAL

The locations and districts of m central facilities have to be determined in such a way that: (a) each user is allocated to exactly one central facility and (b) the access-opportunity of all user locations is as equal as possible when compared with each other:

In applying equation (6.5) for the access-opportunity $_jI_i$ and the concept of the standard deviation the objective function is:

Minimize:

$$Z = ((n\ (n-1))^{-1}\ (n \sum_{j=1}^{n} (\sum_{i=1}^{m}\ _jI_i\ \alpha_{ji})^2 - (\sum_{j=1}^{n} \sum_{i=1}^{m}\ _jI_i\ \alpha_{ji})^2\ ))^{1/2} \qquad \ldots.(7.1)$$

subject to:

$$\alpha_{ji} = \begin{cases} 0 \\ 1 \end{cases} \qquad \text{for all } i,\ i=1,\ldots,m \quad \text{for all } j,\ j=1,\ldots,n \qquad \ldots.(7.2)$$

$$\sum_{i=1}^{m} \alpha_{ji} = 1 \qquad \text{for all } j,\ j=1,\ldots,n \qquad \ldots.(7.3)$$

$$d_{ji} \geq 0 \qquad \text{for all } i,\ i=1,\ldots,m \quad \text{for all } j,\ j=1,\ldots,n \qquad \ldots.(7.4)$$

MODEL SE-6: LOCATIONS AND SPHERES OF INFLUENCE FOR USER-ORIENTED POINTS OF EQUALIZED POTENTIAL

The locations and spheres of influence of m central facilities have to be determined in such a way that: (a) each user is considered to be allocated to each central facility and (b) the access-opportunity at all user locations is as equal as possible when compared with each other:

In applying equation (6.5) for the access-opportunity $_jI_i$ and the concept of the standard deviation the objective function is:

Minimize:

$$Z = ((n\ (n-1))^{-1}\ (n \sum_{j=1}^{n} (\sum_{i=1}^{m}\ _jI_i)^2 - (\sum_{j=1}^{n} \sum_{i=1}^{m}\ _jI_i)^2\ ))^{1/2} \qquad \ldots.(8.1)$$

subject to:

$$d_{ji} \geq 0 \qquad \text{for all } i,\ i=1,\ldots,m \quad \text{for all } j,\ j=1,\ldots,n \qquad \ldots.(8.2)$$

MODEL SE-7: LOCATIONS AND DISTRICTS FOR RADIAL POINTS

The locations and districts of an unknown number of central facilities have to be determined in such a way that: (a) each user is allocated to at least one central facility which is not farther away than a maximum allowable distance d^+ and (b) the number of central facilities is minimized:

Minimize:

$$Z = \sum_{i=1}^{m} X_i \qquad \text{....(9.1)}$$

subject to:

$$\sum_{i=1}^{m} X_i \, \alpha_{ij} \geq 1 \qquad \text{for all } j,\ j=1,\dots,n \qquad \text{....(9.2)}$$

$$X_i = \begin{cases} 0 \\ 1 \end{cases} \qquad \text{for all } i,\ i=1,\dots,m \qquad \text{....(9.3)}$$

$$\alpha_{ij} = \begin{cases} 0 \\ 1 \end{cases} \qquad \begin{array}{l} \text{for all } i,\ i=1,\dots,m \\ \text{for all } j,\ j=1,\dots,n \end{array} \qquad \text{....(9.4)}$$

$$d_{ij}^{\beta} \, \alpha_{ij} \leq d^+ \qquad \begin{array}{l} \text{for all } i,\ i=1,\dots,m \\ \text{for all } j,\ j=1,\dots,n \end{array} \qquad \text{....(9.5)}$$

$$d_{ij} \geq 0 \qquad \begin{array}{l} \text{for all } i,\ i=1,\dots,m \\ \text{for all } j,\ j=1,\dots,n \end{array} \qquad \text{....(9.6)}$$

MODEL SE-8: LOCATIONS AND DISTRICTS FOR CONSTRAINED RADIAL POINTS

The locations and districts of m central facilities have to be determined in such a way that: (a) a user is not allocated to more than one central facility and (b) the sum of users is maximized that can be reached within a maximum allowable distance d^+.

Maximize:

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} r_j \, \alpha_{ij} \qquad \text{....(10.1)}$$

subject to:

$$\alpha_{ij} = \begin{cases} 0 \\ 1 \end{cases} \qquad \begin{array}{l} \text{for all } i,\ i=1,\dots,m \\ \text{for all } j,\ j=1,\dots,n \end{array} \qquad \text{....(10.2)}$$

$$\sum_{i=1}^{m} \alpha_{ij} \leq 1 \qquad \text{for all } j,\ j=1,\dots,n \qquad \text{....(10.3)}$$

$$d_{ij}^{\beta} \, \alpha_{ij} \leq d^+ \qquad \begin{array}{l} \text{for all } i,\ i=1,\dots,m \\ \text{for all } j,\ j=1,\dots,n \end{array} \qquad \text{....(10.4)}$$

$$d_{ij} \geq 0 \qquad \begin{array}{l} \text{for all } i,\ i=1,\dots,m \\ \text{for all } j,\ j=1,\dots,n \end{array} \qquad \text{....(10.5)}$$

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7. FOOTNOTES

(1) See e.g.: TEITZ (1968), MARKS ET AL. (1970), REVELLE ET AL. (1970).

(2) For an application of this approach in the context of fire station location consult SCHILLING (1976, pp. 26-40 and pp. 88-109) and REVELLE ET AL. (1977).

(3) See e.g.: WAGNER (1968), SCHEURWATER (1976), SHEPPARD (1974).

(4) For a detailed discussion of dimensions of models see HARRIS (1961, 1967) or LOWRY (1965).

(5) See BENTELE & MUELLER-TRUDRUNG (1970) or BUNGE (1970, pp. 80-91) for a review.

(6) See DROSNESS & LUBIN (1966), MORRILL (1967), MORRILL & EARICKSON (1968), MORRIL ET AL. (1970), EARICKSON (1970, pp. 8-39 and pp. 55-58), SHANNON ET AL. (1969), WEISS & GREENLICK (1970), SCHULTZ (1970), WEISS ET AL. (1971), ABERNATHY & SCHREMS (1971), DE VISE (1973), SHUMAN ET AL. (1973, pp. 123f), ROTHMEL & HAMMER (1974), PYLE & LAUER (1975).

(7) This principle may be regarded as a specific aspect of ZIPF's (1949, pp. 5-8) more general definition of the 'principle of least effort'.

(8) For a further discussion of this issue the reader is referred to: SYMONS (1971, 1973), MORRILL & SYMONS (1974, 1977), BANERJI & FISHER (1974, pp. 178-180), KOLESAR & WALKER (1972, pp. 2-8), MORRILL (1974), MCALLISTER (1976).

(9) See BACH (1978, pp. 52-98, pp. 246-253) for a mathematical programming formulation as well as methods of solution for each of the following nine models for the location-problem.

(10) See BACH (1978, pp. 99-118, pp. 254-269) for a mathematical programming formulation as well as numerical and graphical methods of solution for each of the following four models for the allocation-problem.

(11) A more detailed discussion of the ten models for the location-allocation-problem is presented in BACH (1978, pp. 119-226). For computer-programs for each of these models - except model SE-2 consult BACH & ZOLLO (1975), BACH & KRUEGER (1976a, 1976b), BACH & KNABE (1977), BACH & SIEMON (1977).

(12) The mathematical programming formulations for some of these extensions are presented in BACH (1978, pp. 234-237).

(13) This extension is not applicable to models SE-7 and SE-8.

(14) A detailed discussion of these four strategies is presented in BACH (1978, pp. 230-234, pp. 237-243).

(15) For dynamic aspects in models for the location-allocation-problem consult also: SCOTT (1971), SHEPPARD (1974), SCHEURWATER (1976, pp. 24-27, p. 34f).

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CENTRAL LOCATIONS — USERS TAKEN INTO CONSIDERATION		DISTRICTS						
		ALL USERS HAVE TO BE TAKEN INTO CONSIDERATION						NOT ALL USERS HAVE TO BE TAKEN INTO CONSIDERATION/ DISTRICTS POSSIBLY OVER-LAP
		DISTRICTS DO NOT OVERLAP		DISTRICTS DO OVERLAP		DISTRICTS POSSIBLY OVERLAP		
		MODEL E-1 POINTS OF EQUAL DISTANCE	MODEL E-2 POINTS OF EQUAL POTENTIAL	MODEL E-3 ATTRACTIVENESS- AND DISTANCE-DEPENDENT SPHERES	NO CRITERIA FOR ALLOCATION	MODEL E-4 (SUBSIDIARY)	MODEL E-1 (SUBSIDIARY)	MODEL E-4 POINTS OF MAXIMUM ALLOWABLE DISTANCE
ALL	MODEL S-1 (S-2, S-3) CENTRAL POINT	MODEL SE-1						
ALL	MODEL S-4 CENTER POINT						MODEL SE-2	
ALL	MODEL S-5 FACILITY-ORIENTED POINT OF POTENTIAL	MODEL SE-3						
ALL	MODEL S-5 (SUBSIDIARY)	MODEL SE-3 /VARIANT 1		MODEL SE-3 /VARIANT 2				
ALL	MODEL S-6 USER-ORIENTED POINT OF POTENTIAL		MODEL SE-4					
ALL	MODEL S-7 USER-ORIENTED POINT OF EQUALIZED POTENTIAL		MODEL SE-5		MODEL SE-6			
ALL	MODEL S-8 RADIAL POINT					MODEL SE-7		
NOT ALL	MODEL S-9 CONSTRAINED RADIAL POINT							MODEL SE-8

ALLOCATION-PROBLEM

LOCATION-ALLOCATION-PROBLEM

LOCATION-PROBLEM

LOCATION-ALLOCATION-PROBLEM

FIGURE 1: Systematic of models for the location-problem, allocation-problem and location-allocation-problem.

PRINCIPLE OF SPATIAL BEHAVIOUR		LOCATIONAL GOALS	MODELS FOR THE LOCATION-ALLOCATION-PROBLEM	
MINIMIZATION OF FRICTION OF SPACE	WITH NO INFLUENCE ON THE VOLUME OF CONSUMPTION OF CENTRAL SERVICES	MINIMIZATION OF THE COLLECTIVE FRICTION OF SPACE	SE-1	LOCATIONS AND DISTRICTS FOR CENTRAL POINTS
		⇕	SE-2	LOCATIONS AND DISTRICTS FOR CENTER POINTS
		EQUALIZATION OF THE INDIVIDUAL FRICTION OF SPACE	SE-7	LOCATIONS AND DISTRICTS FOR RADIAL POINTS
	WITH INFLUENCE ON THE VOLUME OF CONSUMPTION OF CENTRAL SERVICES	MAXIMIZATION OF THE USAGE OF CENTRAL FACILITIES	SE-8	LOCATIONS AND DISTRICTS FOR CONSTRAINED RADIAL POINTS
			SE-3	LOCATIONS AND DISTRICTS FOR FACILITY-ORIENTED POINTS OF POTENTIAL
		⇕	SE-3	/VARIANT 1: LOCATIONS AND DISTRICTS FOR FACILITY-ORIENTED POINTS OF EQUALIZED POTENTIAL
		EQUALIZATION OF THE USAGE OF CENTRAL FACILITIES	SE-3	/VARIANT 2: LOCATIONS AND SPHERES OF INFLUENCE FOR FACILITY-ORIENTED POINTS OF EQUALIZED POTENTIAL
MAXIMIZATION OF POTENTIAL		MAXIMIZATION OF COLLECTIVE ACCESS OPPORTUNITY OF LOCATIONS OF USERS	SE-4	LOCATIONS AND DISTRICTS FOR USER-ORIENTED POINTS OF POTENTIAL
		⇕	SE-5	LOCATIONS AND DISTRICTS FOR USER-ORIENTED POINTS OF EQUALIZED POTENTIAL
		EQUALIZATION OF INDIVIDUAL ACCESS OPPORTUNITY OF LOCATIONS OF USERS	SE-6	LOCATIONS AND SPHERES OF INFLUENCE FOR USER-ORIENTED POINTS OF EQUALIZED POTENTIAL

FIGURE 2: Models for the location-allocation-problem and their relationship to principles of spatial behavior and locational goals.